Math 314 May 2008 — Homework 2

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May 5, 2008

1. Prove the following fact: For a Markov chain with transition matrix P and initial distribution $\boldsymbol{u}_0$, the probability distribution for the states at time n, $\boldsymbol{u}_n$ is given by
$$\boldsymbol{u}_n = P^n \boldsymbol{u}_0.$$
(Hint: Use induction on n and the decomposition of the event $X_n = s_i$ in terms of unions and intersections that we used in class.)

2. Let $\{X_0, X_1, \ldots\}$ be a Markov chain with transition matrix P.
   (a) Define $Y_n = X_{2n}$. Show that $\{Y_0, Y_1, \ldots\}$ is a Markov Chain with transition matrix P^2. (Hint: What property defines a Markov chain?)
   (b) Generalize the above result to taking every k^{th} value of $\{X_0, X_1, \ldots\}$.

3. Consider the Markov Chain with nine states and the following transition matrix:
$$P = \begin{bmatrix} B & 0 & A \\ A & B & 0 \\ 0 & A & B \end{bmatrix}$$
made up of the 3×3 blocks
$$A = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1/2 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}.$$
   (a) Draw the transition graph. (Hint: try a 3×3 grid of states.)
   (b) Is this chain irreducible? What is the minimum number of steps necessary to return to a state?
   (c) Is this chain periodic or aperiodic?

4. Consider a chessboard occupied by a single king. At every turn, he chooses one of the possible squares to which he can move, uniformly at random. Describe the resulting Markov chain. How many states? Is it irreducible or aperiodic? What if the king is replaced by a bishop? A knight?

5. Consider a markov chain with two states and transition matrix

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}.$$

   (a) Draw the transition graph.
   (b) Show that the chain is *not* irreducible.
   (c) What happens to X_n as $n \to \infty$

6. Find the stationary distributions of the Markov chains from problems 3 and 5, using a computer program if necessary.