1. For a random variable $Y$, prove that (a) $E[aY + b] = aE[Y] + b$ and (b) $V(aY + b) = a^2V(Y)$.

**Solution:** Using the definition of expectation,

$$E[aY + b] = \sum_y (ay + b)p(y)$$

$$= \sum_y ay p(y) + \sum_y bp(y)$$

$$= a \sum_y p(y) + b \sum_y p(y)$$

By the definition of expectation, $\sum_y yp(y) = E[Y]$. By the first axiom of probability, $1 = P(S) = \sum_y p(y)$ so

$$E[aY + b] = aE[Y] + b$$

For the variance, using the definition directly

$$V(aY + b) = E \left[ (aY + b - E[aY + b])^2 \right]$$

$$= E \left[ (aY + b - aE[Y] - b)^2 \right]$$

$$= E \left[ (aY - aE[Y])^2 \right]$$

$$= E \left[ a^2(Y - E[Y])^2 \right]$$

$$= a^2E \left[ (Y - E[Y])^2 \right] = a^2V(Y)$$

where the second and fifth equalities use part (a) and the final equality is the definition of variance again.

2. A vicious disease is sweeping the MCSP department. 15% of the department are affected.
   In order to determine what the disease might be, the Centers for Disease Control need to examine three randomly selected people who have the disease. In order to get random
selection, we apply the following procedure: select a member of the department at random from the official roster; check their left nostril for green slime; if they have slime, send them off to the CDC.

What is the probability that we will have to check more than 10 nostrils to get our three subjects for examination? What is the chance that we will get supremely lucky and only have to nasally probe three people?

**Solution:** The event that we have to check more than 10 nostrils is the same as the event where the number of successes in our first ten trials is 0, 1 or 2. The number of successes in 10 trials is a binomial random variable with \( n = 10 \) and \( p = 0.15 \). So, if \( A \) is the event of having to check more than 10 nostrils

\[
P(A) = p(0) + p(1) + p(2)
= 0.1969 + 0.3474 + 0.2759 = 0.8202
\]

The event \( B \) of only having to swab three noses is getting three successes in three trials, so using a binomial random variable with \( n = 3 \) and \( p = 0.15 \)

\[
P(B) = p(3) = \binom{3}{3} (0.15)^3 (0.85)^0 = 0.003375.
\]

3. Let \( Y \) be a binomial random variable with \( n \) trials and probability of success \( p \). Show that for any number \( y_0 \) with \( 0 \leq y_0 \leq n \), \( P(Y = y_0) \) is maximized when \( p = y_0/n \). (This means that if we do observe exactly \( y_0 \) successes, then \( y_0/n \) is in some sense our best guess for \( p \).)

**Solution:** To maximize \( P(y_0) = \binom{n}{y_0} p^{y_0} (1 - p)^{n-y_0} \) with respect to \( p \), we can ignore the binomial coefficient since it doesn’t involve \( p \). Since the natural log is an increasing function, maximizing \( \ln P(y_0) \) is the same as maximizing \( P(y_0) \).

\[
\ln P(y_0) \propto \ln p^{y_0} (1 - p)^{n-y_0} = y_0 \ln p + (n - y_0) \ln(1 - p)
\]

so taking the derivative with respect to \( p \) and equating to zero to find a critical point

\[
\frac{y_0}{p} - \frac{n - y_0}{1 - p} = 0 \Rightarrow (1 - p)y_0 = (n - y_0)p
\]

\[
\Rightarrow p = \frac{y_0}{n}
\]
4. Using only the three axioms of probability theory

\[(i) \forall A \subset S, P(A) \geq 0\]
\[(ii) P(S) = 1\]
\[(iii) \forall A \subset S, \forall B \subset S, A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)\]

prove that \(\forall A \subset S, P(\bar{A}) = 1 - P(A)\).

**Solution:** By the definition of the complement, \(A \cap \bar{A} = \emptyset\) and \(A \cup \bar{A} = S\). Then by axiom (ii), \(1 = P(S) = P(A \cup \bar{A}) = P(A) + P(\bar{A})\) where the last equality uses axiom (iii). Rearranging gives \(P(\bar{A}) = 1 - P(A)\).

5. The 37 students in Dr. Neil’s MA 214 class were asked how many people in their immediate families had graduated from college. The data is summarized in the table below. If the random variable \(X\) is the response of a randomly selected student, find the mean \(E[X]\) and the variance \(V(X)\).

<table>
<thead>
<tr>
<th>response</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>5</td>
<td>9</td>
<td>11</td>
<td>10</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Solution:** The probability distribution of \(X\) is
\[
\begin{array}{c|cccccc}
  x & 0 & 1 & 2 & 3 & 4 & 5 \\
  p(x) & 5/37 & 9/37 & 11/37 & 10/37 & 1/37 & 1/37 \\
\end{array}
\]

Computing using the definition of expectation, \(E[X] = \sum_x xp(x) = 0 \cdot 5/37 + \cdots + 5 \cdot 1/37 = 1.892\) and \(V(X) = E[(X - E[X])^2] = (0 - 1.892)^2 \cdot 5/37 + \cdots + (5 - 1.892)^2 1/37 = 1.394\).

6. A continuous random variable \(Y\) with density function
\[
f(y) = \begin{cases} 
  \frac{my^{m-1}}{\alpha} e^{-y^m/\alpha}, & y \geq 0 \\
  0, & \text{otherwise}
\end{cases}
\]

is said to have a **Weibull distribution**. It is frequently used to model the occurrence of extreme events.

(a) What is the usual name of a Weibull distribution with \(m = 1\)? Find the mean and variance of such a random variable.
(b) Find the mean and variance of a Weibull distribution with \( m = 2 \). Be sure to show your work.

**Solution:** By part (c), \( E[Y] = \sqrt{\alpha} \Gamma \left( \frac{3}{2} \right) \) and

\[
V(Y) = \alpha \left( \Gamma (2) + \Gamma \left( \frac{3}{2} \right)^2 \right).
\]

(c) Derive, don’t just state, the mean and variance of a Weibull distributed random variable in terms of \( \alpha \) and \( m \).

**Solution:** We’ll compute the mean and variance directly by computing the moments.

\[
E[Y^n] = \int_0^\infty \frac{my^{m+n-1}}{\alpha} e^{-y^{m}/\alpha} \, dy
\]

\[
= \int_0^\infty (\alpha u)^{n/m} e^{-u} \, du
\]

\[
= \alpha^{n/m} \int_0^\infty u^{n/m} e^{-u} \, du
\]

\[
= \alpha^{n/m} \Gamma \left( 1 + \frac{n}{m} \right)
\]

Where we used the substitution \( u = y^{m}/\alpha \). Then, \( E[Y] = \alpha^{1/m} \Gamma \left( 1 + \frac{1}{m} \right) \) and

\[
V(y) = E[Y^2] - E[Y]^2
\]

\[
= \alpha^{2/m} \Gamma \left( 1 + \frac{2}{m} \right) - \left( \alpha^{1/m} \Gamma \left( 1 + \frac{1}{m} \right) \right)^2
\]

\[
= \alpha^{2/m} \left( \Gamma \left( 1 + \frac{2}{m} \right) - \Gamma \left( 1 + \frac{1}{m} \right)^2 \right)
\]

7. The time to process an order at the local pharmacy is exponentially distributed with mean 6 minutes. If 25 customers place orders at the pharmacy in one week, what is the probability that 10 or fewer customers need to wait more than 6 minutes?

**Solution:** The number of people waiting more than six minutes \( X \) is a binomial random variable with \( n = 25 \) and where success is waiting more than six minutes. If
$Y$ is the waiting time for a single customer, then $p$ the probability of success is

\[
p = P(Y > 6) = \int_{6}^{\infty} \frac{1}{6} e^{-y/6} \, dy = \int_{1}^{\infty} e^{-u} \, du = \frac{1}{e}
\]

For a binomial random variable with $n = 25$ and $p = \frac{1}{e}$, $P(X \leq 10) = 0.0154$.

8. Using the moment generating function for the gamma distribution with parameters $\alpha$ and $\beta$, find the first three moments about the origin, $\mu_1'$, $\mu_2'$ and $\mu_3'$.

**Solution:** The moment generating function is $m(t) = (1 - \beta t)^{-\alpha}$. Taking three derivatives

\[
m'(t) = -\alpha(1 - \beta t)^{-\alpha-1}(-\beta) = \alpha \beta (1 - \beta t)^{-\alpha-1}
\]

\[
m''(t) = \alpha \beta (-\alpha - 1)(1 - \beta t)^{-\alpha-2}(-\beta) = \alpha (\alpha + 1) \beta^2 (1 - \beta t)^{-\alpha-2}
\]

\[
m'''(t) = \alpha (\alpha + 1) \beta^2 (-\alpha - 2)(1 - \beta t)^{-\alpha-3}(-\beta) = \alpha (\alpha + 1) (\alpha + 2) \beta^3 (1 - \beta t)^{-\alpha-3}
\]

and setting $t = 0$ gives $\mu_1' = \alpha \beta$, $\mu_2' = \alpha (\alpha + 1) \beta^2$ and $\mu_3' = \alpha (\alpha + 1) (\alpha + 2) \beta^3$.

9. (a) Show that

\[
\mu_2 = E[(Y - \mu)^2] = E[Y^2] - \mu^2 = \mu_2' - (\mu_1')^2.
\]

This formula relates the second central moment $\mu_2$ to the first two moments about the origin $\mu_2'$ and $\mu_1'$.

**Solution:**

\[
\mu_2 = E[(y - \mu)^2] = E[Y^2 - 2\mu Y + \mu^2] = E[Y^2] - 2\mu E[Y] + \mu^2 = E[Y^2] - 2\mu^2 + \mu^2 = E[Y^2] - \mu^2 = \mu_2' - (\mu_1')^2
\]

(b) Derive a similar formula relating $\mu_3$ to the first three moments about the origin.
Solution:

\[
\mu_3 = E[(y - \mu)^3] = E[Y^3 - 3\mu Y^2 + 3\mu^2 Y - \mu^3] \\
= E[Y^3] - 3\mu E[Y^2] + 3\mu^2 E[Y] - \mu^3 \\
= \mu'_3 - 3\mu'_1\mu'_2 + 2(\mu'_1)^3
\]

10. Two methods are available for teaching students to integrate: the Carrot and the Stick. The Carrot is more successful than the stick, with 10% of students taught by the Carrot failing to learn integration, while 20% of students taught with the Stick fail to learn integration. However, Carrots are much harder to use than Sticks and so only 1/3 of students receive instruction using the Carrot.

Given that a student has failed to learn how to integrate, what is the probability that they were taught using a Stick?

Solution: With the events \( F \): failed to learn integration, \( C \): learned using the carrot and \( S \): learned using the stick, we are given that \( P(C) = 1/3 \), \( P(S) = 2/3 \), \( P(F|C) = 0.1 \) and \( P(F|S) = 0.2 \). Using Bayes’ Rule,

\[
P(S|F) = \frac{P(F|S)P(S)}{P(F|S)P(S) + P(F|C)P(C)} \\
= \frac{0.2(2/3)}{0.2(2/3) + 0.1(1/3)} \\
= \frac{4/3}{4/3 + 1/3} = \frac{4}{5}
\]

11. A fair \( n \)-sided die is tossed \( n \) times. What is the probability that the numbers seen were \( 1, \ldots, n \) in any order? Plot your answer as a function of \( n \) for reasonable values of \( n \).

Solution: On every roll, we must roll a number that hasn’t been rolled before. For the first roll, this happens with probability 1. For the second roll, the probability is \((n - 1)/n\) that we roll one of the \( n - 1 \) remaining unrolled options. Continuing this pattern gives us

\[
\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n} \cdot \frac{1}{n}
\]

which simplifies to

\[
\frac{n!}{n^n} = \frac{(n-1)!}{n^{n-1}}
\]

That function is plotted here on a log-plot for \( n \) ranging from 2 to 12.
12. A certain MA 313 student has clairvoyantly determined that this exam will be composed of twelve out of twenty possible problems. But, in studying those twenty problems, she is only able to solve fifteen correctly. If I do in fact make this exam by choosing twelve problems at random from those twenty, what is the probability that your psychic classmate scores 100% on this exam?

**Solution:** This is a hypergeometric probability with \( N = 20, n = 12 \) and \( r = 15 \). Using the hypergeometric distribution,

\[
p(12) = \frac{\binom{15}{12} \binom{5}{0}}{\binom{20}{12}} = .00361
\]