Baire Category and Nowhere Differentiability for Feasible Real Functions*

Josef M. Breutzmann[1], David W. Juedes[2], and Jack H. Lutz[3]

[1] Dept. Math/CS/Physics, Wartburg College, Waverly IA 50677 U.S.A.
breutzma@wartburg.edu
[2] School of EE & CS, Ohio University, Athens, OH 45701, U.S.A.
juedes@ohiou.edu
[3] Dept. of Comp. Sci., Iowa State University, Ames, IA 50011 U.S.A.
lutz@cs.iastate.edu

Abstract. A notion of resource-bounded Baire category is developed for the class $P_{\mathcal{C}[0,1]}$ of all polynomial-time computable real-valued functions on the unit interval. The meager subsets of $P_{\mathcal{C}[0,1]}$ are characterized in terms of resource-bounded Banach-Mazur games. This characterization is used to prove that, in the sense of Baire category, almost every function in $P_{\mathcal{C}[0,1]}$ is nowhere differentiable. This is a complexity-theoretic extension of the analogous classical result that Banach proved for the class $\mathcal{C}[0,1]$ in 1931.

1 Introduction

Baire category and Lebesgue measure provide a structural framework to classify the relative sizes of infinite sets in various spaces. In the context of complexity theory, the space that we are most familiar with is the space of all languages, *i.e.*, the Cantor space. Unfortunately, since most sets of languages of interest (P, NP, etc.) inside of the Cantor space are countable, classical versions of category and measure cannot classify the relative sizes of these sets in any nontrivial way. To remedy this situation, computable versions of category were investigated by Mehlhorn [18] and Lisagor [10], and resource-bounded versions of measure and category were developed by Lutz [11–14], Fenner [6, 7], Mayordomo [16, 17], Allender and Strauss [1], Strauss [23], and others. Resource-bounded category and measure have been used successfully to examine the structure of complexity classes in a variety of contexts [2, 4, 24, etc.]. The recent surveys by Lutz [15] and Ambos-Spies and Mayordomo [3] provide an overview of work in this area.

In contrast to classical complexity theory, the complexity theory of real functions [9] works primarily in the space $\mathcal{C}[0,1]$ consisting of all continuous functions over the closed interval $[0,1]$. As in the Cantor space, all countable subsets of $\mathcal{C}[0,1]$ are small (meager, measure 0) in the senses of Baire category and Lebesgue measure. Hence, these classical theories cannot classify sets of computable real

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functions in any nontrivial way. To remedy this situation, we develop a resource-bounded version of Baire category in $\mathcal{C}[0,1]$ and use it to investigate the distribution of differentiability in $P_{\mathcal{C}[0,1]}$, the class of all polynomial-time computable, continuous functions over the closed interval $[0,1]$.

Let $\mathcal{ND} = \{f \in \mathcal{C}[0,1] \mid f \text{ is nowhere differentiable}\}$, where $\mathcal{C}[0,1]$ is the space of all continuous functions $f : [0,1] \longrightarrow \mathbb{R}$. In the nineteenth century, Weierstrass [25] exhibited a function $f \in \mathcal{ND}$. Subsequently, many other such functions have been shown to exist [26, etc.]. In 1931 Banach [5] proved that $\mathcal{ND}$ is a *comeager* subset of $\mathcal{C}[0,1]$ in the sense of Baire category. That is, $\mathcal{C}[0,1] - \mathcal{ND}$ is meager. Banach's result implies the result of Weierstrass, since $\mathcal{C}[0,1]$ is not meager. However, Banach's result says more — it says that any subset of $\mathcal{C}[0,1]$ that is not meager contains a nowhere differentiable function. Hence, the existence of nowhere differentiable functions with various properties can be shown by direct application of the category result.

As mentioned above, $P_{\mathcal{C}[0,1]}$ is a countable, and hence meager subset of $\mathcal{C}[0,1]$. Hence, Banach's result cannot be used to demonstrate the existence of a polynomial-time computable real valued function that is nowhere differentiable. Indeed, Banach's result leaves the possibility that no polynomial-time computable function is nowhere differentiable. However, this is not the case. As shown by Ko [9], certain well-known nowhere differentiable functions are, in fact, polynomial-time computable. Indeed, related results for computable functions were shown much earlier in the work by Myhill [19] and Pour-El and Richards [21]. Here we show that $\mathcal{ND}$ is comeager in $P_{\mathcal{C}[0,1]}$, a result that implies both Banach's original result [5] and Ko's later result [9] for the polynomial-time computable functions.

The paper is structured as follows. In section 2, we give the necessary preliminary notation and definitions from real analysis, Baire category, and the complexity theory of real functions. In section 3, we define a resource-bounded Baire Category for $\mathcal{C}[0,1]$. The central definition is that of the p-meager sets in $\mathcal{C}[0,1]$. As we show, every p-meager set is meager in the classical sense. Following the work of Lutz [12, 13], Fenner [6, 7], and Strauss [23], we also show that the class of p-meager sets forms an ideal of "small" sets. That is, the p-meager sets satisfy the following conditions: (i) subsets of p-meager sets are p-meager, (ii) the p-meager sets are closed under finite unions, (iii) the p-meager sets are closed under effective countable unions, (iv) for each function $f \in P_{\mathcal{C}[0,1]}$, $\{f\}$ is p-meager, and (v) $P_{\mathcal{C}[0,1]}$ is not p-meager. In addition, we give a characterization of the p-meager sets in terms of resource-bounded Banach-Mazur games in $\mathcal{C}[0,1]$. In section 4, we use this characterization to prove our main result, namely, that $\overline{\mathcal{ND}[0,1]}$ is p-meager. This implies that the set of polynomial-time computable functions that have a derivative at some point $x \in [0,1]$ is a negligibly small subset of $P_{\mathcal{C}[0,1]}$. The proofs of all technical results in section 3 are omitted from this extended abstract.

2 Preliminaries

We begin by presenting the necessary notation and definitions from real analysis, Baire category, and complexity theory of real functions. For a more detailed presentation, see Rudin [22], Oxtoby [20], or Ko [9]. To begin, let $\mathcal{C}[0,1]$ be the set of continuous real valued functions on the compact domain $[0,1]$. Given any functions f and g in $\mathcal{C}[0,1]$, the distance between f and g is

$$d(f,g) = ||f-g|| = \sup_{x\in[0,1]} |f(x)-g(x)|.$$

It is well-known that $\mathcal{C}[0,1]$ along with the associated distance function d form a complete metric space.

Since $\mathcal{C}[0,1]$ is a complete metric space, we will sometimes call a function $f \in \mathcal{C}[0,1]$ a *point*. For $r > 0$, the *neighborhood of radius* r about the function f is the set $N_r(f)$ containing all functions g such that $d(f,g) < r$. Let S be a subset of $\mathcal{C}[0,1]$. A function f is a *limit point* of S if for every $r > 0$ there exists a $g \neq f$ such that $g \in N_r(f) \cap S$. If every function f that is a limit point of S is contained in S, then S is *closed*.

Given a sequence $f_0, f_1, \ldots, f_n, \ldots$ of functions in $\mathcal{C}[0,1]$, the limit of this sequence is defined point-wise. That is, if the sequence $\{f_n(x)\}$ converges for each $x \in [0,1]$, then the limit of $\{f_n\}$ is the function f defined by $f(x) = \lim_{n\longrightarrow\infty} f_n(x)$. Since $\mathcal{C}[0,1]$ is a compact space, the limit of a sequence of continuous functions is also continuous.

If there is an $r > 0$ such that $N_r(f) \subseteq S$, then the function f is an *interior point* of S. If every function $f \in S$ is an interior point of S, then S is *open*. If every function in $\mathcal{C}[0,1]$ is contained in S or a limit point of S (or both), then S is *dense* in $\mathcal{C}[0,1]$.

According to [8], a set S is *nowhere dense* in $\mathcal{C}[0,1]$ if and only if for each open set O, $O \cap S$ is not dense in O. Equivalently, a set S is nowhere dense if, for every open set O, there exists an open set $O' \subseteq O$ such that $O' \cap S = \emptyset$. A set S is *meager* (a set of first category) in $\mathcal{C}[0,1]$ if it is a countable union of a family nowhere dense sets. A set S is *nonmeager* (a set of second category) if it is not meager. A set S is *comeager* (residual) if its complement is meager.

Following the work of Ko [9], we use the dyadic rational numbers $\mathbb{D} = \{m \cdot 2^{-n} | m \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$ as finite approximations to real numbers. Because the dyadic rational numbers are dense in $\mathbb{R}$, it is possible to define the topology of $\mathcal{C}[0,1]$ in terms piece-wise linear functions with dyadic rational endpoints. A function $f \in \mathcal{C}[0,1]$ is a *piece-wise linear function with dyadic rational endpoints* if there exist points $a_0 = 0 < a_1 < a_2 \ldots < a_n = 1 \in \mathbb{D}$ such that $f(a_i) \in \mathbb{D}$ and for $a_i < x < a_{i+1}$, $f(x) = f(a_i) + \frac{f(a_{i+1})-f(a_i)}{a_{i+1}-a_i}(x-a_i)$. A *basic open set* O is a set $O = N_d(f)$, where $d \in \mathbb{D} \cup \{\infty\}$, $d > 0$, and f is a piece-wise linear function with dyadic rational endpoints. It is well-known that a set $S \subseteq \mathcal{C}[0,1]$ is nowhere dense if and only if for every basic open set O there exists a basic open set $O' \subseteq O$ such that $O' \cap S = \emptyset$.

Here we are primarily interested in $P_{\mathcal{C}[0,1]}$, the set of functions in $\mathcal{C}[0,1]$ that are feasibly computable. Using Theorem 2.22 of Ko [9, p. 59], we define $P_{\mathcal{C}[0,1]}$

to be the set of functions $f \in \mathcal{C}[0,1]$ where there exists a sequence of piece-wise linear functions $\{f_n\}$ with dyadic rational endpoints and a polynomial-function m such that

(i) for each $n \in \mathbb{N}$ and $0 \le i \le 2^{m(n)}$, $f_n(\frac{i}{2^{m(n)}}) \in \mathbb{D}$,
(ii) For each n and $0 \le i < 2^{m(n)}$, $|f_n(\frac{i}{2^{m(n)}}) - f_n(\frac{i+1}{2^{m(n)}})| \le 2^{-n}$,
(iii) for each $n \in \mathbb{N}$ and $x \in [0,1]$, $|f_n(x) - f(x)| \le 2^{-n}$,
(iv) the polynomial function $m(n) : \mathbb{N} \to \mathbb{N}$ is computable in time $p(n)$, and the function $\psi : \mathbb{D} \times \mathbb{N} \to \mathbb{D}$ defined by $\psi(\frac{i}{2^{m(n)}}, n) = f_n(\frac{i}{2^{m(n)}})$ is computable in time $q(m(n)+n)$. Here, both p and q are polynomials.

Finally, we define $DTIME(n^c)_{\mathcal{C}[0,1]}$ to be the set of all functions $f \in P_{\mathcal{C}[0,1]}$ satisfying the above conditions and the condition that $p(n) + q(m(n)+n) = O(n^c)$.

3 Resource-Bounded Baire Category in C[0,1]

Let $\mathcal{B}$ be the set of all basic open sets. Then, a set $X \subseteq \mathcal{C}[0,1]$ is nowhere dense if there exists a function $\alpha : \mathcal{B} \to \mathcal{B}$ such that for every basic open set $x \in \mathcal{B}$, $\alpha(x) \subseteq x$ and $\alpha(x) \cap X = \emptyset$. Such a function α "testifies" that X is nowhere dense. Intuitively, such a function α takes a basic open set and creates a refinement of that basic open set that misses X. Similarly, a set $X = \bigcup_{i=1}^{\infty} X_i$ is meager if there exists a function $\alpha' : \mathbb{N} \times \mathcal{B} \to \mathcal{B}$ such that the function $\alpha'_i(x) = \alpha'(i,x)$ testifies that X_i is nowhere dense.

Since each basic open set has a finite binary representation, a natural approach to resource-bounded Baire category on $\mathcal{C}[0,1]$ might be to require that α' be computable in some resource-bound, e.g., X is Δ-meager if there exists a function α' that testifies that X is meager and α' is computable in the resources given by Δ. Unfortunately, this natural approach does not allow for a reasonable notion of category inside of $P_{\mathcal{C}[0,1]}$ because a basic open set's finite binary representation may need to be exponentially large. To remedy this situation, we instead examine functions that refine segments of basic open sets. We begin by presenting the necessary definitions.

Definition 1. *A* neighborhood component code *is a 6-tuple $\kappa = \langle n, r, a, b, c, d\rangle$ such that $n, a, b \in \mathbb{N}$, $c, d \in \mathbb{Z}$, $r \in \mathbb{Z} \cup \{\infty\}$, and $0 \le \frac{a}{2^n} < \frac{b}{2^n} \le 1$. The* neighborhood component *corresponding to a neighborhood component code $\kappa = (n, a, b, c, d, r)$ is the set $N(\kappa)$ consisting of all functions $f \in \mathcal{C}[0,1]$ such that for all $x \in [\frac{a}{2^n}, \frac{b}{2^n}]$, $\varphi_\kappa(x) - 2^r < f(x) < \varphi_\kappa(x) + 2^r$, where $\varphi_\kappa(x) = \frac{d-c}{b-a}(x - \frac{a}{2^n}) + \frac{c}{2^n}$.*

Each basic open set can be viewed as a sequence of consistent neighborhood components. This notion of consistency is defined as follows.

Definition 2. *A neighborhood component code $\kappa_1 = \langle n_1, r_1, a_1, b_1, c_1, d_1\rangle$* meets *$\kappa_2 = \langle n_2, r_2, a_2, b_2, c_2, d_2\rangle$ if $n_1 = n_2$, $r_1 = r_2$, $b_1 = a_2$, and $d_1 = c_2$. A* neighborhood code *on an interval $[a,b]$ is a finite sequence $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_l)$ of neighborhood component codes such that κ_i meets κ_{i+1} for all $1 \le i < l$, $\frac{a_1}{2^n} = a$,* *and* $\frac{b_l}{2^n} = b$*, where* n *is the common first component of all the* κ_i*. The* neighborhood corresponding to a neighborhood code $\boldsymbol{\kappa}$ *on an interval* $[a,b]$ *is the set* $N(\boldsymbol{\kappa}) = \bigcap_{i=1}^{n} N(\kappa_i)$.

It is easy to see that every basic open set is the neighborhood corresponding to some neighborhood code $\boldsymbol{\kappa}$ on $[0,1]$. In order to define the meager sets, we will need a notion of the refinement of a neighborhood.

Definition 3. *A* refinement *of a neighborhood component code* $\kappa = \langle n,r,a,b,c,d\rangle$ *is a neighborhood code* $\boldsymbol{\kappa} = (\kappa_1,\ldots,\kappa_l)$ *on* $[\frac{a}{2^n},\frac{b}{2^n}]$ *such that* $N(\boldsymbol{\kappa}) \subseteq N(\kappa)$ *and* $r_1 < r$*.*

Now, let $\mathcal{N}_0$ be the set of all neighborhood component codes, and let $\mathcal{N}$ be the set of all neighborhood codes. A *constructor* is a function $\gamma : \mathcal{N}_0 \longrightarrow \mathcal{N}$ such that

(i) $(\forall\kappa \in \mathcal{N}_0)\gamma(\kappa)$ is a refinement of κ, and
(ii) γ is *consistent* in the sense that if κ_1 meets κ_2 then the right hand component of $\gamma(\kappa_1)$ meets the left hand component of $\gamma(\kappa_2)$.

Given a constructor α, it is natural to extend the application of α from individual neighborhood component codes to full neighborhood codes. Given a constructor α, define $\overline{\alpha} : \mathcal{N} \longrightarrow \mathcal{N}$ by $\overline{\alpha}((\kappa_1,\ldots,\kappa_l)) = \overline{(\alpha(\kappa_1),\ldots,\alpha(\kappa_l))}$, where $\overline{(\boldsymbol{\kappa_1},\ldots,\boldsymbol{\kappa_l})}$ is the vector containing the individual components (in order) of the vectors $\boldsymbol{\kappa_1},\ldots,\boldsymbol{\kappa_l}$.

Such constructors can be used to testify that sets are nowhere dense.

Theorem 1. *Let* α *be a constructor and let* $X \subseteq \mathcal{C}[0,1]$*. If it is the case that* $N(\overline{\alpha}(\boldsymbol{\kappa})) \cap X = \emptyset$ *for every neighborhood code* $\boldsymbol{\kappa}$ *that corresponds to a basic open set, then* X *is nowhere dense.*

Proof. This is immediate from the fact that α is a constructor.

It is not known whether the converse is true, *i.e.*, that every nowhere dense set has a constructor that testifies that it is nowhere dense. Nevertheless, this approach provides a reasonable notion of category in $P_{\mathcal{C}[0,1]}$.

To define a notion of resource-bounded Baire Category on $P_{\mathcal{C}[0,1]}$, we apply resource bounds to our constructors. A constructor γ is computable in polynomial time if the function $\hat{\gamma} : \mathcal{N}_0 \times \mathbb{N} \longrightarrow \mathcal{N}_0 \cup \{\perp\}$ defined by

$$\hat{\gamma}(\kappa, i) = \begin{cases} \kappa_i & \text{if } 1 \le i \le l \\ \perp & \text{otherwise,} \end{cases}$$

where $\gamma(\kappa) = (\kappa_1,\ldots,\kappa_l)$, is computable in time polynomial in $|\kappa| + |i|$. Note that we assume that $\kappa = \langle n,r,a,b,c,d\rangle$ is encoded with n and r represented in unary with an additional sign bit for r. It follows that $|\kappa| \ge n + |r|$.

An indexed constructor is a function $\alpha' : \mathbb{N} \times \mathcal{N}_0 \longrightarrow \mathcal{N}$ such that $\alpha'(i, \circ)$ is a constructor for each $i \in \mathbb{N}$. An indexed constructor α' is computable in polynomial time if $\widehat{\alpha}'(i,\kappa,j)$ is computable in time bounded by a polynomial in $|i| + |\kappa| + |j|$. We will use indexed constructors to define a notion of meager sets in $P_{\mathcal{C}[0,1]}$.

Definition 4. *A set X is* p-meager *if* $X = \bigcup_{i=1}^{\infty} X_i$ *and there exists a polynomial-time computable indexed constructor* α' *such that* $\alpha'(i, \circ)$ *testifies that* X_i *is nowhere dense. A set* X *is* p-comeager *if* $\overline{X} = \mathcal{C}[0,1] - X$ *is* p*-meager. A set* X *is* meager in $P_{\mathcal{C}[0,1]}$ *if* $X \cap P_{\mathcal{C}[0,1]}$ *is* p*-meager. A set* X *is* comeager in $P_{\mathcal{C}[0,1]}$ *if* $\overline{X}$ *is meager in* $P_{\mathcal{C}[0,1]}$.

Example 1. The set $X = \{f \in \mathcal{C}[0,1] | f(1/4) = f(3/4)\}$ is p-meager. Hence, X is meager in $P_{\mathcal{C}[0,1]}$.

As shown in the previous example, certain simple sets of functions can be shown to be p-meager using Definition 4. In some cases, it is desirable to work with a modified definition that uses a slightly restricted notion of an indexed constructor. We say that a constructor $\alpha : \mathcal{N}_0 \to \mathcal{N}$ is *q-bounded* if there exists a polynomial q such that for every $\kappa \in \mathcal{N}_0$, if $\alpha(\kappa) = (\kappa_1, \ldots)$ and $\kappa_1 = \langle n, r, a, b, c, d \rangle$ then $n \leq q(|r|)$. An indexed constructor $\alpha' : \mathbb{N} \times \mathcal{N}_0 \to \mathcal{N}$ is q-bounded if there exists a single polynomial q such that $\alpha'(i, \circ)$ is q-bounded for every i. Notice that it is easy to prove that $X = \bigcup_{i=1}^{\infty} X_i$ is p-meager if and only if there exists a polynomial-time computable q-bounded indexed constructor α' such that $\alpha'(i, \circ)$ testifies that X_i is nowhere dense.

The rationale for using this modified definition of indexed constructors lies in that fact that constructors implicitly define real valued functions. To see this, begin with a basic open set O and iteratively apply some constructor α. If α is computable, the single function in the intersection of the closures of these basic open sets is a computable function. However, if α is computable in polynomial time such a construction may not produce a polynomial-time computable function unless α is q-bounded.

We next give an equivalent definition of the p-meager sets in terms of resource-bounded Banach-Mazur games.

3.1 Resource-Bounded Banach-Mazur Games

It is well-known [20] that Baire category can be characterized in terms of a two person game of perfect information called the Banach-Mazur game. In this context, a Banach-Mazur game is a two person game where the players alternately restrict a set of viable functions. The game begins with $\mathcal{C}[0,1]$, the set of all continuous functions on [0,1], and a set $X \subseteq \mathcal{C}[0,1]$. Player I begins by producing a basic open set B_1. Player II then produces a basic open set $B_2 \subseteq B_1$ whose radius decreases by at least one half. The game continues forever with player I and player II alternately restricting the resulting basic open set. The result of the game is the single function f contained in the intersection of the closure of the basic open sets produced during each round of the game. Player I wins if $f \in X$. Player II wins if $f \notin X$. A set X is meager if and only if there is a strategy so that player II always wins on X.

Here we characterize the p-meager sets in terms of Banach-Mazur games where the two players are indexed constructors. Let α and β be indexed constructors, and let $N(\boldsymbol{\kappa})$ be a basic open set. The kth round of the Banach-Mazur game

$[\alpha, \beta; X]$ consists of applying $\overline{\alpha}(k, \circ)$ to $\boldsymbol{\kappa}$ and then applying $\overline{\beta}(k, \circ)$ to $\overline{\alpha}(k, \boldsymbol{\kappa})$. The game starts with $\boldsymbol{\kappa} = (\langle 0, \infty, 0, 1, 0, 0 \rangle)$. The neighborhood corresponding to $\boldsymbol{\kappa}$ is the neighborhood of radius ∞ about the piecewise linear function $f(x) = 0$. This neighborhood contains all of $\mathcal{C}[0,1]$. Now, define $\boldsymbol{\kappa}_i$ as follows.

$$\boldsymbol{\kappa}_0 = \boldsymbol{\kappa} = (\langle 0, \infty, 0, 1, 0, 0 \rangle)$$
$$\boldsymbol{\kappa}_{2i+1} = \overline{\alpha}(i, \boldsymbol{\kappa}_{2i}), \ \boldsymbol{\kappa}_{2i+2} = \overline{\beta}(i, \boldsymbol{\kappa}_{2i+1}).$$

The result of the game $[\alpha, \beta; X]$ is the unique function $f \in \bigcap_{i=0}^{\infty} \overline{N}(\boldsymbol{\kappa}_i)$, where $\overline{N}(\boldsymbol{\kappa}_i)$ is the closure of the neighborhood corresponding to $\boldsymbol{\kappa}_i$. Player I wins if $f \in X$, and player II wins if $f \notin X$.

It is straightforward to prove that if both player I and player II are polynomial-time computable indexed constructors that $f \in P_{\mathcal{C}[0,1]}$.

Theorem 2. *Let α and β be polynomial-time computable q-bounded indexed constructors. Then, the unique function f that is the result of the game $[\alpha, \beta; X]$ is an element of $P_{\mathcal{C}[0,1]}$.*

Similarly, if $f \in P_{\mathcal{C}[0,1]}$, then f is the result of some Banach-Mazur game.

Theorem 3. *If $f \in P_{\mathcal{C}[0,1]}$, then there exist polynomial-time computable q-bounded indexed constructors α and β such that f is the result of the game $[\alpha, \beta; X]$.*

If player II (β) to wins the game $[\alpha, \beta; X]$ for all possible α, then this is equivalent to X being a p-meager set.

Theorem 4. *Let $X \subseteq \mathcal{C}[0,1]$. The following are equivalent.*

a. X is p-meager.
b. There exists a polynomial-time q-bounded indexed constructor β such that player II wins the game $[\alpha, \beta; X]$ for all indexed constructors α.

3.2 Basic Results

We end this section with a collection of basic results concerning the p-meager sets. Following previous work on resource-bounded measure and category [12, 7, 23, etc.], we show that the p-meager sets in $P_{\mathcal{C}[0,1]}$ satisfy those conditions expected for a class of small sets, i.e., the p-meager sets are closed under subset, finite union, and appropriate countable union; each singleton $\{f\}$ for $f \in P_{\mathcal{C}[0,1]}$ is p-meager; and $P_{\mathcal{C}[0,1]}$ is not p-meager. We begin by giving a definition of an appropriate countable union of p-meager sets.

Definition 5. *A p-union of p-meager sets is a set X such that there exists a polynomial-time indexed constructor α and a family of sets $\{X_i | i \in \mathbb{N}\}$ such that*

(i) $X = \bigcup_{i=1}^{\infty} X_i$.

(ii) *For each* i*, the indexed constructor* α_i *defined by* $\alpha_i(j, \boldsymbol{\kappa}) = \alpha(\langle i, j\rangle, \boldsymbol{\kappa})$ *testifies that* X_i *is p-meager.*

Theorem 5. *The following conditions concerning the p-meager sets hold.*

(i) *If* X *is p-meager and* $Y \subseteq X$*, then* Y *is p-meager.*
(ii) *If* X *and* Y *are p-meager, then* $X \cup Y$ *is p-meager.*
(iii) *If* X *is a p-union of p-meager sets, then* X *is p-meager.*
(iv) *If* $f \in P_{\mathcal{C}[0,1]}$*, then* $\{f\}$ *is p-meager.*

Theorem 6. *(Baire Category Theorem)* $P_{\mathcal{C}[0,1]}$ *is not p-meager.*

4 Nowhere Differentiability in $P_{\mathcal{C}[0,1]}$

We now present a nontrivial application of the theory of resource-bounded Baire category in $P_{\mathcal{C}[0,1]}$. Here we examine the distribution of differentiability in $P_{\mathcal{C}[0,1]}$. As we show in Theorem 7 below, the set of nowhere differentiable functions, $\mathcal{ND}[0,1]$, is p-comeager and hence comeager in $P_{\mathcal{C}[0,1]}$. This result implies the classical result of Banach and existence of nowhere differentiable functions in $P_{\mathcal{C}[0,1]}$. The proof of Theorem 7 requires the following technical lemma.

Lemma 1. *If* $\kappa = \langle n, r, a, b, c, d\rangle$ *is a neighborhood component code with central segment* L *and* L' *is any segment* $\overline{P_1P_2}$ *within* κ*, i.e.,* $P_1 = (\frac{a}{2^n}, y_1)$ *and* $P_2 = (\frac{b}{2^n}, y_2)$ *with* $|y_1 - \frac{c}{2^n}| < 2^r$ *and* $|y_2 - \frac{d}{2^n}| < 2^r$*, then the slopes* m *and* m' *of* L *and* L' *respectively satisfy* $|m - m'| < 2^{n+r+1}$.

Theorem 7. $\mathcal{ND}[0,1]$ *is p-comeager.*

Proof. We define a polynomial-time computable clocked constructor γ with which player II can force the result of a Banach-Mazur game to be an element of $\mathcal{ND}[0,1]$. Hence $\overline{\mathcal{ND}[0,1]}$ is p-meager and $\mathcal{ND}[0,1]$ is p-comeager. In our construction, $\gamma(i, \circ)$ does not depend on the parameter i. Hence, we write $\gamma(\kappa)$ for $\gamma(i, \kappa)$.

Given a neighborhood component code $\kappa = \langle n, r, a, b, c, d\rangle$ we define $\gamma(\kappa)$ as follows: first select the least $n' \in \mathbb{N}$ such that $n' > n$, $n' > |r|$, and

$$2^{n'+r} > 8|r| + 4. \quad (1)$$

Second, select the greatest $r' \in \mathbb{Z}$ such that $r' < r$ and

$$2^{n'+r'+1} < |r|. \quad (2)$$

These choices of n' and r' depend only on n and r and can be done consistently (in polynomial-time) across all of [0,1].

The constructor $\gamma(\kappa)$ creates $l = (b - a) \cdot 2^{n'-n}$ subintervals of width $2^{-n'}$. The structure of new neighborhood components within the subintervals depends on the slope $m = \frac{d-c}{b-a}$ of the central segment of κ. There are two cases.

Case 1: $|m| > 2|r|+1$. In this case, since the slope is already steeper than $|r|$, we attempt to keep the slope of the central segment in each subinterval as close to m as possible. Define $\hat{\gamma}(\kappa, i) = \kappa'_i = \langle n', r', a'_i, b'_i, c'_i, d'_i \rangle$ as follows. Let n' and r' be as given earlier. For $1 \leq i \leq l$, $a'_i = a \cdot 2^{n'-n} + (i-1)$, $b'_i = a'_i + 1$, $c'_1 = c \cdot 2^{n'-n}$, and $d'_l = d \cdot 2^{n'-n}$. Note that $\frac{a'_1}{2^{n'}} = \frac{a}{2^n}$, $\frac{b'_l}{2^{n'}} = \frac{b}{2^n}$, $\frac{c'_1}{2^{n'}} = \frac{c}{2^n}$, and $\frac{d'_l}{2^{n'}} = \frac{d}{2^n}$. For $2 \leq i \leq l$, let $c'_i = d'_{i-1} = \varphi_1(i-1)$, where $\varphi_1(i) = c'_1 + \lceil m \cdot i \rceil$. Since $n' > |r|$ and $r' < r$, it follows that these subintervals lie within κ. Moreover, the slope of the central segment for each subinterval is

$$m' = \frac{d'_i - c'_i}{b'_i - a'_i} = \varphi_1(i) - \varphi_1(i-1) = \lceil m \cdot i \rceil - \lceil m(i-1) \rceil.$$

Since $x \leq \lceil x \rceil < x + 1$, it is easy to show that $m - 1 < m' < m + 1$. It follows that $|m'| > 2|r|$.

By Lemma 1, the slope m'' for any segment L'' within $\kappa'_i = \langle n', r', a'_i, b'_i, c'_i, d'_i \rangle$ will differ from m', the slope of the central segment by at most $|m'' - m'| < 2^{n'+r'+1}$. Since $2^{n'+r'+1} < |r|$, we have $|m'' - m'| < |r|$. Since $|m'| > 2|r|$, it follows that $|m''| > |r|$.

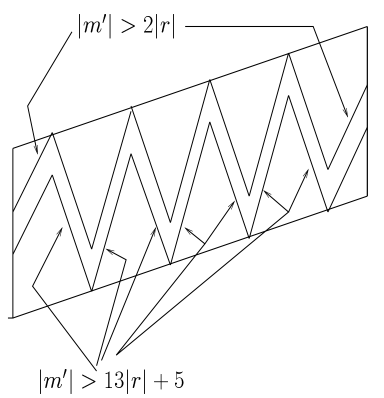

Fig. 1. A closer view of Case 2

Case 2: As seen in Figure 1, if $|m| \leq 2|r|+1$, to make the slopes of the refinement steeper than $|r|$ we introduce a sawtooth neighborhood inside of κ so that the absolute value of the slope of the central segment of each component is at least $2|r|$. As before, the original neighborhood component code is broken down into $l = (b-a) \cdot 2^{n'-n}$ subintervals of width $2^{-n'}$. For each $1 \leq i \leq l$, let $a'_i = a \cdot 2^{n'-n} + (i-1)$ and $b'_i = a'_i + 1$. As before, set $c'_1 = c \cdot 2^{n'-n}$ and $d'_l = d \cdot 2^{n'-n}$. This provides consistency with the neighboring segments. For $2 \leq i \leq l-1$, let

$$d'_{i-1} = c'_i = \varphi_2(i) = \begin{cases} c'_1 + \lfloor m(i-1) + 2^{n'+r} - 2^{n'+r'} \rfloor & \text{if } i \text{ is even} \\ c'_1 + \lceil m(i-1) - 2^{n'+r} + 2^{n'+r'} \rceil & \text{if } i \text{ is odd.} \end{cases}$$

Notice that this definition places the neighborhood $\gamma(\kappa)$ inside of κ.

Now let's examine the slopes of the central segments in each subinterval. When $2 \le i \le l-1$ and i is odd, the slope of the central segment of κ'_i is

$$m' = \frac{d'_i - c'_i}{b'_i - a'_i} = d'_i - c'_i = \lfloor m \cdot i + 2^{n'+r} - 2^{n'+r'} \rfloor - \lceil m(i-1) - 2^{n'+r} + 2^{n'+r'} \rceil$$
$$= \lfloor m \cdot i - 2^{n'+r'} \rfloor - \lceil m(i-1) + 2^{n'+r'} \rceil + 2^{n'+r+1}$$

The final equality holds because $2^{n'+r}$ is an integer whenever $n' > |r|$. Moreover, because $x - 1 < \lfloor x \rfloor \le x$ and $x \le \lceil x \rceil < x + 1$, it is easy to show that $m + 2^{n'+r+1} - 2^{n'+r'+1} - 2 < m' \le m + 2^{n'+r+1} - 2^{n'+r'+1}$. Since $2^{n'+r} > 8|r| + 4$, $2^{n'+r'+1} < |r|$, and $|m| \le 2|r| + 1$, it follows that $m' > 13|r| + 5$. Similarly, we can show that $m' < -13|r| - 5$ when $2 \le i \le l-1$ and i is even.

When $i = 1$, the slope of the central segment of κ'_1 is

$$m' = \frac{d'_1 - c'_1}{b'_1 - a'_1} = d'_1 - c'_1 = \lfloor m + 2^{n'+r} - 2^{n'+r'} \rfloor = \lfloor m - 2^{n'+r'} \rfloor + 2^{n'+r}.$$

Because $r' < r$, it is easy to show that $m - 1 + 2^{n'+r-1} < m' \le m + 2^{n'+r}$. Since $|m| \le 2|r| + 1$ and $2^{n'+r-1} > 4|r| + 2$, it follows that $m' > 2|r|$. Similarly, we can show that $|m'| > 2|r|$ when $i = l$.

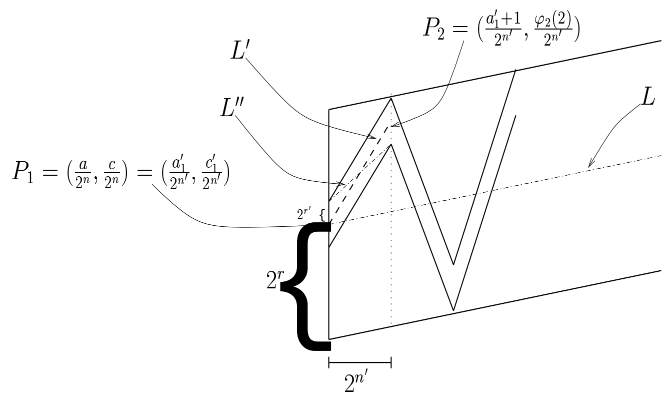

Fig. 2. A closer view of Case 2

Consider a segment L'' with slope m'' inserted into one of the neighborhoods for these l subintervals, e.g., see Figure 2. If we apply Lemma 1 to the neighborhood κ'_i, we have $|m'' - m'| < 2^{n'+r+1}$. Since $|m'| > 2|r|$ and $2^{n'+r+1} < |r|$, it follows that $|m''| > |r|$.

Thus, in either case, we have that any segment lying "within" a neighborhood component code κ'_i of $\gamma(\langle n, r, a, b, c, d \rangle)$ has slope that exceeds r in absolute value. We now complete the proof through the use of two claims.

Claim. γ is polynomial-time computable.

Proof. Given n, r, we can find n' and r' satisfying (1) and (2) in polynomial-time in the size of n and r. In addition, we can compute the near linear functions $\varphi_1(i)$ and $\varphi_2(i)$ in polynomial-time in the size of the input and $|i|$, and hence we can compute $\kappa_i = \hat{\gamma}(\kappa, i)$ in polynomial time.

Claim. If $f(x) : [0,1] \to \mathbb{R}$ is the result of a Banach Mazur game in which player II uses strategy γ, then $f \in \mathcal{ND}[0,1]$. Hence, $\overline{\mathcal{ND}[0,1]}$ is p-meager and $\mathcal{ND}[0,1]$ is p-comeager.

Proof. Let $x \in [0,1]$, $\epsilon > 0$, $M > 0$ be given, and let $f(x)$ be the result of the a Banach-Mazur game in which player II used strategy γ. Since $n, |r| \to \infty$, at some point during the game there must have been a neighborhood code $\boldsymbol{\kappa}$ given to player II with a component code κ such that x lies in $\kappa_i' = \gamma(\kappa, i) = \langle n', r', a_i', b_i', c_i', d_i' \rangle$ with $2^{-n'} < \epsilon$, $|r| > M$, and $\frac{a_i'}{2^{n'}} \leq x \leq \frac{b_i'}{2^{n'}}$.

Now, let $P_1 = (\frac{a_i'}{2^{n'}}, f(\frac{a_i'}{2^{n'}}))$, $P = (x, f(x))$, and $P_2 = (\frac{b_i'}{2^{n'}}, f(\frac{b_i'}{2^{n'}}))$. By the construction of γ, the slope of m of $\overline{P_1P_2}$ must satisfy $|m| > |r| > M$. By the triangle inequality, if m_1 is the slope of $\overline{P_1P}$ and m_2 is the slope of $\overline{PP2}$, one of m_1 or m_2 must satisfy $|m_i| > |r| > M$. Hence, P_1 or P_2 provides a point which yields a difference quotient whose absolute value exceeds M at x. So, f fails to be differentiable at x since M and ϵ were arbitrary. Further, x was arbitrary, and so $f \in \mathcal{ND}[0,1]$.

Since player II forced f into $\mathcal{ND}[0,1]$ via γ, $\mathcal{ND}[0,1]$ is p-comeager. This completes the proof of Theorem 7.

Corollary 1. *(**Banach [5]**) $\mathcal{ND}[0,1]$ is comeager.*

Proof. This follows from that fact that every p-meager set is meager.

Corollary 2. *(**Ko [9]**) There exists a function $f \in P_{\mathcal{C}[0,1]}$ that is nowhere differentiable.*

Proof. This follows from Theorems 6 and 7.

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